On Electromagnetic Spinors and Quantum Theory

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The Maxwell theory is related to the Dirac theory by two heuristic arguments based on electromagnetic spinors. First, given the condition to have spinors, if the free field Maxwell's equations are satisfied, then the spinors satisfy the Weyl equations. Second, a sufficient condition for electromagnetic spinors directly gives the Weyl equations with the usual substitutions for momentum and energy. It is proposed that the heuristic nature of the first argument is related to the idea of a point having a more general symmetry than a sphere.

That Maxwell's equations may be written in Dirac form was seen by Darwin [1] in 1928, and more recently has captured the attention of several authors [2–5]. Given the fields

$$\boldsymbol{\Phi}_{\mathrm{R}} = \boldsymbol{\sigma} \cdot (\boldsymbol{H} - i\,\boldsymbol{E}), \quad \boldsymbol{\Phi}_{\mathrm{L}} = \boldsymbol{\sigma} \cdot (\boldsymbol{H} + i\,\boldsymbol{E}), \tag{1}$$

the sourceless Maxwell equations may be written as

$$\left[-\,i(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) - i\,\frac{\partial}{\partial t}\right] \boldsymbol{\Phi}_{\mathrm{R}} = 0, \tag{2}$$

$$\left[-\,i(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) + i\,\frac{\partial}{\partial t}\right] \boldsymbol{\Phi}_{\mathrm{L}} = 0. \tag{3}$$

If the fields are subjected to the condition that associates them with spinor fields, then there is an heuristic connection to the Weyl equations.

It is known that the rotational symmetries of a spinor can be modeled by the curious notion of a vector with zero length. Following Cartan [6], consider a Euclidian space E_3 with a vector $\boldsymbol{x} = (x_1, x_2, x_3)$, generally with complex components, that satisfies

$$x_1^2 + x_2^2 + x_3^2 = 0. \tag{4}$$

Then we may define two numbers ξ_0, ξ_1 by

$$x_1 = \xi_0^2 - \xi_1^2, \quad x_2 = i(\xi_0^2 + \xi_1^2), \quad x_3 = -\,2\,\xi_0\,\xi_1, \tag{5}$$

which are consistent with taking

$$\xi_0 = \pm\sqrt{\frac{x_1 - i\,x_2}{2}}, \quad \xi_1 = \pm\sqrt{\frac{-x_1 - i\,x_2}{2}}. \tag{6}$$

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Under rotations the two component vector $\boldsymbol{\xi} = (\xi_0, \xi_1)$ transforms as a spinor. The relationship between the zero length vector and its associated spinor may also be expressed as

$$(\boldsymbol{\sigma} \cdot \boldsymbol{x})\,\boldsymbol{\xi} = 0. \tag{7}$$

Taking the fields (1) as having zero length at each point of space, we have

$$\boldsymbol{\Phi}_{\mathrm{L}}\,\boldsymbol{\varphi}_{\mathrm{L}} = 0, \quad \boldsymbol{\Phi}_{\mathrm{R}}\,\boldsymbol{\varphi}_{\mathrm{R}} = 0, \tag{8}$$

where $\boldsymbol{\varphi}_{\mathrm{L}}$, $\boldsymbol{\varphi}_{\mathrm{R}}$ are the spinor fields associated with $\boldsymbol{\Phi}_{\mathrm{L}}$, $\boldsymbol{\Phi}_{\mathrm{R}}$. Then

$$\begin{aligned}&\left[-\,i\,(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) - i\,\frac{\partial}{\partial t}\right] \boldsymbol{\Phi}_{\mathrm{R}}\,\boldsymbol{\varphi}_{\mathrm{R}} \\ &= \left\{\left[-\,i\,(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) - i\,\frac{\partial}{\partial t}\right] \boldsymbol{\Phi}_{\mathrm{R}}\right\} \boldsymbol{\varphi}_{\mathrm{R}} \\ &\quad + \boldsymbol{\Phi}_{\mathrm{R}} \left\{\left[-\,i\,(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) - i\,\frac{\partial}{\partial t}\right] \boldsymbol{\varphi}_{\mathrm{R}}\right\} = 0.\end{aligned} \tag{9}$$

Therefore, if $\boldsymbol{\Phi}_{\mathrm{R}}$ satisfies (2), but is otherwise arbitrary, then $\boldsymbol{\varphi}_{\mathrm{R}}$ will of necessity satisfy the Weyl equation

$$\left[-\,i\,(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) - i\,\frac{\partial}{\partial t}\right] \boldsymbol{\varphi}_{\mathrm{R}} = 0. \tag{10}$$

A similar result follows for $\boldsymbol{\varphi}_{\mathrm{L}}$.

The Weyl equations also follow from the definition of a spinor by taking a more traditional point of view. From the identity

$$(\boldsymbol{\sigma} \cdot \boldsymbol{a})\,(\boldsymbol{\sigma} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} + i\,\boldsymbol{\sigma} \cdot (\boldsymbol{a} \wedge \boldsymbol{b}) \tag{11}$$

the condition for spinors may be expressed as

$$\boldsymbol{\Phi}_{\mathrm{L}}\,\boldsymbol{\Phi}_{\mathrm{R}} + \boldsymbol{\Phi}_{\mathrm{R}}\,\boldsymbol{\Phi}_{\mathrm{L}} = 0. \tag{12}$$

Then a sufficient condition for electromagnetic spinors is

$$\boldsymbol{\Phi}_{\mathrm{L}}\,\boldsymbol{\Phi}_{\mathrm{R}} = H^2 + E^2 - 2\,\boldsymbol{\sigma} \cdot (\boldsymbol{E} \wedge \boldsymbol{H}) = \mathscr{E} - \boldsymbol{\sigma} \cdot \boldsymbol{P} = 0 \tag{13}$$

and

$$\boldsymbol{\Phi}_{\mathrm{R}}\,\boldsymbol{\Phi}_{\mathrm{L}} = H^2 + E^2 + 2\,\boldsymbol{\sigma} \cdot (\boldsymbol{E} \wedge \boldsymbol{H}) = \mathscr{E} + \boldsymbol{\sigma} \cdot \boldsymbol{P} = 0, \tag{14}$$

where $\mathscr{E}$, $\boldsymbol{P}$ are the energy and momentum densities. The Weyl equations follow from the substitutions $\mathscr{E} \to i\,\frac{\partial}{\partial t}$, $\boldsymbol{P} \to -\,i\,\boldsymbol{\nabla}$. (Although this discussion is limited to wave mechanics, it is interesting to note the similarity to the second quantized Dirac theory in (8) and (12–14)).

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The heuristic nature of the argument leading to (10) follows from a closer examination of the geometrical model of a spinor. A vector cannot have a zero length with nonzero components. For the constraint (4) to have meaning, it must be expressed as the differential

$$(dx_1)^2 + (dx_2)^2 + (dx_3)^2 = 0. \tag{15}$$

This expression may be contrasted with the familiar metric of isotropic space,

$$(dx_1)^2 + (dx_2)^2 + (dx_3)^2 = (ds)^2. \tag{16}$$

Imagining the differentials to have finite extensions, (15) is numerically false, though (16) is satisfied. But collapsing to a point at the infinitesimal limit, either of these may possibly be true. If (16) holds, a point will have the symmetry of a sphere, 0(3); if (15) holds, a point will have the SU(2) symmetry. As (15) is true only at a point and cannot be reached by a limiting process without numerical impossibility, the differentiation of the fields leading to (10) cannot be taken by a limiting process either. But at the limit, the argument leading to (10) apparently has no rigorous meaning and will be at best heuristic. Still, if (15) is indeed the proper way of describing spinors, there is a pleasant connection between half integral spin and point particles.

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